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Affine connections in modified general relativity with Quantum-Deformed metric tensor

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Through a consistent extension of the minimal length, derived from the noncommutative Heisenberg algebra and the relativistic generalized uncertainty principle (RGUP) that integrates finite gravitational fields into quantum mechanics (QM), quantum-induced revisions are imposed on the spacetime metric along with quantum modifications to general relativity (GR). The proposed quantization scheme is conjectured to remain compatible with conventional GR by introducing additional curvatures on relativistic eight-dimensional tangent bundle, Finsler geometry that generalizes the Riemann geometry. We analyze the symmetry properties of both quantum-deformed metric and affine connections, and examine how parallel transport acquires dependence on second-order derivatives of the tangent covectors, normalized to quantities induced by quantum and gravitational effects. Our analysis shows that the quantum-induced contributions factorize linearly, such that their removal restores the classical geometric structures exactly. We conclude that the quantum-induced corrections endow both quantities with distinct quantum characteristics, particularly in the relativistic regime. At the same time, their fully quantum nature implies that the fundamental metric and the associated 1-form variations must be described by probability distributions, noncommutative structures, or quantum operators.

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1. Introduction

It is widely acknowledged that general relativity (GR) asserts the geometrical nature of the gravitational field.[1] This is represented on a four-dimensional Riemann manifold, which is provided with a symmetric metric tensor and affine (linear) connections.[2] These connections are defined as torsion-free and metric-compatible; hence, they can be fully represented in terms of the metric tensor itself. An affine connection is a geometric construct on a *smooth* manifold that connects neighboring tangent spaces. The tangent vector fields are represented as covariant derivatives on this manifold. Furthermore, the affine connections specify the procedure for conducting parallel transport of tangent vectors on the manifold. In GR, these connections act as the gravitational force fields, while the metric tensor functions as the relevant gravitational potential.[3]

The fundamental conceptual issues in reconciling the principles of quantum mechanics (QM) with GR arise, among others, due to the violation of the principle of relativity and the principle of equivalence at quantum scales.[4–6] Whereas GR, which considers events defined along the world line of a point particle and produces deterministic results, QM prohibits any exact trajectory due to uncertainty and noncommutativity.[7–9] The violation of the equivalence principle occurs within the framework of quantum gravity as a *local* quantum field theory, despite the nonrenormalizability of GR.[10] In this regard, GR conceptualizes spacetime as a four-dimensional differentiable manifold characterized by a clearly defined metric structure. Given that in QM, the notion of measurement is essential to the true essence of physical reality, the utilization of a measuring instrument — which inevitably experiences quantum fluctuations — prevents the measurement of the background spacetime with infinite precision.[11] Any effort to assess the local properties of the background metric results in an unavoidable disruption in spacetime. This conclusion was drawn from Mead's thought experiment: the interplay of the Heisenberg uncertainty principle (HUP) and GR, at the Planck length enforces a limitation on the precision of measuring spacetime structure.[12,13]

One of the most rigorous attempts to reconcile QM with GR dates back to the 1960s.[14] The uncertainty principle in a gravitational field has been proposed as a means to determine the quantitative limitations on the uncertainty of distance measurement in Minkowski spacetime, $\Delta\ell$,[15]

$$\Delta\ell \propto \ell^{1/3}\, \ell_{\mathtt{pl}}^{3/2}, \tag{1}$$

where $\ell_{\mathtt{pl}}$ denotes the Planck length. This work introduces a minimal-length framework in which the inherent uncertainties emerging in the detection of a quantum state are constrained by noncommutative operators governed by the HUP. The latter limits simultaneous measurements but obviously does not incorporate the effects of gravitational fields.[16–19] The recent extended versions of the HUP known as generalized uncertainty principle (GUP) and the relativistic generalized uncertainty principle (RGUP) are also predicted in string theory, loop quantum gravity, doubly special relativity, and various thought experiments.[20–33] Both approaches may be viewed as a revision emerging from the gravitational effects on quantum measurements, which are essential components of the underlying quantum theory. In other words, the RGUP helps explain the origin of the gravitational field and how a particle behaves within it.[26,30,33] Recently, the effects of the minimal-length approach on the line element, the metric tensor, and the geodesics have been analyzed.[34] An additional term that includes the RGUP parameter along with the second-order derivatives of tangent covectors appears in both line element and metric tensor, while in the geodesics several terms with higher-order derivatives are found.[35]

The current approach of utilizing minimal length to integrate quantum-induced modifications on the fundamental metric has recently been expanded within the framework of quantum relativity theory.[16,18,19,33,36–45] In this paper, we emphasize the significant effects that the concept of minimal length has on reevaluating the fundamental metric from a quantum perspective. By extending the four-dimensional manifold M_4 to an eight-dimensional spacetime tangent bundle $TM_8 = TM_4 \otimes M_4$,[46–49] the quantum-induced corrections to the metric tensor can be derived, Sec. 3. In Sec. 2, a brief review of the minimal measurable length is outlined. The quantum-induced corrections to the affine connections on Riemann manifold are discussed in Sec. 4. The parallel transport of a vector on the Riemann manifold is elaborated in Sec. 5. The symmetry properties of the quantized metric tensor and affine connections are discussed in Sec. 6. Section 7 is devoted to conclusions and outlook.

2. Minimal Measurable Length in General Relativity

Approaches including doubly special relativity, quantum gravity, and string theory suggest a fundamental minimal length, which in QM appears as an irreducible position uncertainty induced by gravitational corrections to the Heisenberg principle,[50]

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}\left[1 + \beta(\Delta p)^2 + \beta\langle p\rangle^2\right], \tag{2}$$

where Δx and Δp represent the length and momentum uncertainties, respectively, $\langle p\rangle$ is the momentum expectation value, G is the gravitational constant, c is the speed of light, and $\hbar$ is the Planck constant. The RGUP parameter is $\beta = \beta_0 G/(c^3\hbar) = \beta_0(\ell_{\rm p}/\hbar)^2$, with β_0 a dimensionless numerical constant to be determined from table-top laboratory experiments[51] or recent cosmological observations.[52] For a Lorentz-covariant algebra, d-dimensional deformed noncommutative algebra is conjectured to quantize $(d+1)$-dimensional spacetime.[53] For $d = 3$, Snyder algebra becomes the special case of spacetime quantization.[54] From Jacobi identities, the general commutation relation of position and momentum operators can be written as

$$[x_i, p_j] = i\hbar[(1 + \beta p^2)\delta_{ij} + 2\beta p_i \hat{p}_j), \tag{3}$$

where the indexes $i, j \in \{1, 2, 3\}$. Although the GUP approach has been successfully applied to various physical phenomena,[13,21,25,27,54–56] the space–momentum uncertainties, Eq. (2), and the commutation relation, Eq. (3), for instance, are not Lorentz covariant. One reason for this is the absence of the temporal dimension.[53] Thus, the corresponding minimal measurable length appears to depend on the frame of reference. Furthermore, the generalization of the momentum operator leads to a nonlinear addition law for momenta[57] and a deformation of their dispersion relations.[26,30,33]

To modify the energy regime for potential applications of the RGUP in quantum gravity, relativistic QM, and quantum field theory, we take the stance that the physical position and momentum coordinates are defined using auxiliary 4-vectors x_0^μ, p_0^ν, which are canonically conjugate variables with the commutation relation $[x_0^\mu, p_0^\nu] = i\hbar\eta^{\mu\nu}$, as follows:

$$x^\mu = x_0^\mu \equiv (x_0^0, x_0^i), \tag{4}$$

$$p^\mu = p_0^\mu(1 + \beta p_0^\rho p_{0\rho}) \equiv (p_0^0,\, p_0^i)(1 + \beta p_0^\rho p_{0\rho}), \tag{5}$$

where $\mu, \nu \in \{0, 1, 2, 3\}$ and ρ is a dummy index. Both $x_0^0 = ct$ and $p_0^0 = E/c$ may be taken as physical quantities. We also take into account Snyder algebra in relation to spacetime noncommutativity,[54] which is associated with translational invariance for the total momentum. Therefore, in a composite system of quantum particles functioning as sources of the gravitational field and moving at the same velocity, it is likely that the nonlinear law of momentum addition remains undeformed.[57] With the isotropic property of spacetime and under Snyder algebra,[54] the RGUP approach is proposed as[21,26,30,33]

$$[x^\mu,\, p^\nu] = i\hbar[(1 + \beta p^\rho p_\rho)\eta^{\mu\nu} + 2\beta p^\mu p^\nu]. \tag{6}$$

This relation is Lorentz covariant in four-dimensional spacetime. From the Robertson uncertainty relation,[58] which follows from the Schrödinger uncertainty relation,[59] the position and momentum noncommutation relation, for instance, determines the uncertainties of their measurements,

$$\Delta x^\mu\, \Delta p^\nu \geq \frac{1}{2}|\langle[x^\mu,\, p^\nu]\rangle|. \tag{7}$$

To convert this relationship into curved spacetime, it may be necessary to transform the flat $\eta^{\mu\nu}$ into the curved metric tensor $g^{\mu\nu}$. The mapping $\eta^{\mu\nu} \to g^{\mu\nu}$ is not inherently simple unless local inertial frames are applied and coordinate transformations are feasible. First, let us recall the different possibilities for substituting Minkowski $\eta_{\mu\nu}$ with the curved metric $g_{\mu\nu}$.

Only at a specific point p in a curved spacetime, coordinates can be chosen such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$. This is apparently based on the equivalence principle of GR, which allows for the selection of a coordinate system where the laws of physics are in agreement with those of special relativity. Therefore, at any point p, there exists a coordinate system (known as Riemann normal coordinates) such that $g_{\mu\nu} = \eta_{\mu\nu}$ and the affine connection $\Gamma^{\rho}_{\mu\nu}$ vanishes. On the other hand, with the general coordinate transformation $x^{\alpha} \to x^{\mu}(x)$, one finds that

$$g_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta}. \tag{8}$$

Although this expression is generally applicable, there is no singular global transformation when considering curved spacetime. A specific mathematical relationship between curved and flat metrics can be represented by

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}(p) x^{\alpha} x^{\beta} + \mathcal{O}(x^3). \tag{9}$$

This is Taylor expansion where the zeroth-order is $\eta_{\mu\nu}$, the first-order is zero, and the second-order indicates the curvature. Additionally, tetrads provide another general relationship,

$$g_{\mu\nu}(x) = e^{\alpha}_{\ \mu} e^{\beta}_{\ \nu} \eta_{\alpha\beta}. \tag{10}$$

The various approaches used in the transformation of $\eta_{\mu\nu}$ to $g_{\mu\nu}$, along with their respective expressions and ranges of validity, are detailed in Table 1.

For the present calculations, we implement the most general conversion, Table 1. This allows to derive the uncertainties associated with measurements of position and momentum, Eq. (7), in curved spacetime,

$$\Delta x^{\mu}\, \Delta p^{\nu} \gtrsim \frac{\hbar}{2} [g^{\mu\nu} + \beta (\Delta p)^2 + \beta \langle p \rangle^2 - \beta (\Delta p^{\mu})^2 - \beta (\Delta p^{\nu})^2]. \tag{11}$$

It is important to note that in order to derive the roots for Eq. (11), the indexes on the left-hand side are temporarily disregarded for the purpose of simplicity. To ensure that the quadratic equation, Eq. (11), has real roots for Δp^{ν}, it is essential that

$$(\Delta x^{\mu})^2 \gtrsim \hbar^2 [\beta^2 (\langle p \rangle^2 - (\Delta p^{\mu})^2 - (\Delta p^{\nu})^2) - \beta\, g^{\mu\nu}]. \tag{12}$$

For nearly vanishing higher-orders of β, the RGUP parameter, the uncertainty associated with the minimal measurable length is expressed as

$$\Delta x^{\mu}_{\mathtt{min}} \gtrsim \pm \sqrt{-g^{\mu\nu}}\, \hbar \sqrt{\beta}, \tag{13}$$

Table 1. A summary of the various possibilities for the conversion of the flat metric tensor $\eta_{\mu\nu}$ to curved metric tensor $g_{\mu\nu}$.

Approach	Expression	Validity
Tetrads	$g_{\mu\nu} = e^{\alpha}_{\ \mu} e^{\beta}_{\ \nu} \eta_{\alpha\beta}$	Everywhere
Riemann normal coordinates	$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} x^{\alpha} x^{\beta} + \mathcal{O}(x^3)$	In a small neighborhood
Local inertial frame	$g_{\mu\nu}(p) = \eta_{\mu\nu}$	At a single point p
Coordinate transformation	$g_{\mu\nu} = \frac{\partial x^a}{\partial x^{\mu}} \frac{\partial x^b}{\partial x^{\nu}} \eta_{ab}$	In flat spacetime

where $g^{\mu\nu} := (g^{-1})^{\mu\nu}$ is the contravariant symmetric $(2,0)$ tensor field with $g := \texttt{det}\, g_{\mu\nu}$. From $\Delta x^{\mu}_{\texttt{min}} = \pm\sqrt{-g}\,\sqrt{\beta_0}\,\sqrt{G\hbar/c^3} = \pm\sqrt{-g}\,\sqrt{\beta_0}\,\ell_{\texttt{pl}}$, one concludes that (i) the uncertainty associated with the minimal measurable length is linearly correlated with the Planck length, characterized by the proportionality factor $\pm\sqrt{\beta_0}$, and (ii) there is a spacetime element $\Delta x^{\mu}_{\texttt{min}}$ within which measurements cannot be conducted — meaning, access is prohibited. Assuming that there is no summation over indexes, $\sqrt{-g^{\mu\nu}}$ can be defined as $\sqrt{-\texttt{det}\, g}$. Consequently, we determine that the minimal measurable length $\Delta x^{\mu}_{\texttt{min}} = \pm\sqrt{\beta_0}\ell_{\texttt{p}}$ remains invariant under coordinate transformation:

$$\sqrt{-\texttt{det}\, g}\,\Delta x^{\mu}_{\texttt{min}} = \Delta\tilde{x}^{\mu}_{\texttt{min}}. \tag{14}$$

Due to the principle of relativity and its isotropic nature, no particular direction is favored in GR. Therefore, it appears that Eq. (14) violates this limitation and suggests that $|\Delta x_{\texttt{min}}| = 2\sqrt{\beta_0}\ell_{\texttt{p}}$ is uniform across all spacetime coordinates.

To further clarify this new result, it is important to note that in GR, the coordinates do not possess any physical dimensions that are predetermined. The coordinates are essentially arbitrary. This implies that the physical interpretation of distance or volume, for instance, is contingent upon the choice of coordinates, which can differ from one coordinate system to another. The Jacobian determinant, a scalar quantity, ensures an invariant transformation from one coordinate system to another. The metric tensor — in principle — plays a similar role. Fortunately, the result shown in Eq. (13) implies that the value of $|\Delta x_{\texttt{min}}|$ does not rely on the choice of spacetime coordinates in which the minimal length uncertainty is defined. From Eq. (13), one concludes that the coordinates in GR should not be solely restricted to x^{μ}_{0} as detailed in Eq. (4). With the minimal measurable length, Eq. (4) can be formulated as

$$x^{\mu} = x^{\mu}_{0} - \Delta x^{\mu}_{\texttt{min}} = ((x^{0}_{0} - |\Delta x_{\texttt{min}}|),\, (x^{i}_{0} - |\Delta x_{\texttt{min}}|)), \tag{15}$$

so that $x^{\mu} \to 0$ if $x^{\mu}_{0} \to \Delta x^{\mu}_{\texttt{min}}$. Then, the uncertainty in the length measurement in GR is given as

$$\Delta x^{\mu} = \Delta x^{\mu}_{0} - 2\sqrt{\beta_0}\ell_{\texttt{p}}. \tag{16}$$

At a relevant scale, the length measurement in GR is seemingly determined by quantum-mechanical principles; that is to say, it is no longer arbitrarily small.

Moreover, according to the solutions of Eq. (11), the uncertainty in momentum is denoted by

$$\Delta p^{\nu} \lesssim \frac{1}{\hbar\,\beta}[\Delta x^{\mu} \pm ((\Delta x^{\mu})^2 - \beta\,\hbar^2\, g^{\mu\nu})^{1/2}]. \tag{17}$$

Equation (17) may be approximated as

$$\Delta p^{\nu} \lesssim \frac{1}{\hbar\,\beta}\left[\Delta x^{\mu} \pm \left(\Delta x^{\mu} + \frac{1}{2}\,\frac{(\Delta x^{\mu}_{\texttt{min}})^2}{\Delta x^{\mu}}\right)\right]. \tag{18}$$

This suggests that the uncertainty in momentum measurement is limited as follows:

$$-\frac{\Delta x^{\mu}}{\hbar\,\beta}\,\frac{1}{2}\left(\frac{\Delta x^{\mu}_{\mathtt{min}}}{\Delta x^{\mu}}\right)^{2} \lesssim \Delta p^{\nu} \lesssim \frac{2\,\Delta x^{\mu}}{\hbar\,\beta}\left[1+\frac{1}{4}\left(\frac{\Delta x^{\mu}_{\mathtt{min}}}{\Delta x^{\mu}}\right)^{2}\right]. \tag{19}$$

For vanishing $(\Delta x^{\mu}_{\mathtt{min}}/\Delta x^{\mu})^2$, one finds that Δp^{ν} is conditioned to

$$\Delta p^{\nu} \lesssim \frac{2}{\hbar\,\beta}\,\Delta x^{\mu}. \tag{20}$$

From Eqs. (5) and (16), the uncertainty in Δp^{ν}_0 is then given as

$$\Delta p^{\nu}_0 \lesssim \frac{2}{\hbar\,\beta}\,\frac{\Delta x^{\mu}_0 - 2\sqrt{\beta_0}\ell_{\mathtt{p}}}{1+\beta\, g^{0\rho}\,\Delta|p_0|^2}. \tag{21}$$

By expanding upon the fundamental principles of QM, specifically the HUP, and incorporating the influences of the gravitational field, it is possible to associate the minimal measurable length $\Delta x_{\mathtt{min}}$ with the Planck constant. Utilizing $\sqrt{-|g|}$, which represents the absolute value of the determinant of the Jacobian matrix J, the action in GR turns to be invariant under diffeomorphism. Consequently, the length measurement also maintains its invariance when coordinates are modified. The results described in this section, if validated, grant GR in the relativistic context the following distinct quantum properties:

- Spacetime is no longer regarded as smooth and continuous; rather, it exhibits roughness and discretization, indicating the probable existence of an inaccessible spacetime element whose volume is represented by $\Delta x_{\mathtt{min}}$.
- Due to the noncommutativity and uncertainties associated with coordinates and momentum, events are likely to manifest in abrupt (jump) transitions with unpredictable and nondeterministic results.
- Measurements no longer exhibit precision or coherence.

For an approximate evaluation of the minimal length and maximal momentum as derived from the RGUP approach,[26,30,33] let us recall Eqs. (13) and (20), respectively. For the RGUP parameter $\beta = \beta_0 G/\hbar c^3$,[52] it is found that

$$\Delta x_{\mathtt{min}} \gtrsim \ \pm\sqrt{-\mathtt{det}g}\,\sqrt{\beta_0}\,\sqrt{\frac{\hbar G}{c^3}} \equiv \pm\sqrt{-\mathtt{det}g}\,\sqrt{\beta_0}\,\ell_{\mathtt{pl}}, \tag{22}$$

$$\Delta p_{\mathtt{max}} \lesssim \ \pm\, 2\,\frac{\sqrt{-\mathtt{det}g}}{\beta_0^2}\,\sqrt{\frac{\hbar c^3}{G}} \equiv \pm\frac{\sqrt{-\mathtt{det}g}}{\beta_0}\,p_{\mathtt{pl}}, \tag{23}$$

where $\pm$ is not merely of mathematical significance; when combined with $\sqrt{-\mathtt{det}g}$, it ensures an overall positive value. The parameter β_0 is a dimensionless RGUP parameter that can be estimated either empirically or through observational methods.[51,60] In a curved spacetime framework, $\sqrt{-\mathtt{det}g}$ represent scalar density with a weight of ± 1, depending on the signature of the smooth, symmetric, rank-2 tensor field $g^{\mu\nu}$ on the four-dimensional differentiable Lorentzian manifold M_4. Depending on the quantitative estimation for β_0, the numerical values for the minimal length and maximal momentum are dictated by the physical constants gravitational constant G, the Planck constant $\hbar$, and the speed of light c or Planck units; length $\ell_{\mathtt{pl}}$ and momentum $m_{\mathtt{pl}}$, respectively. According to expressions (22) and (23), it is evident that both Planck quantities are seemingly modified in curved spacetime and as a result of the RGUP ansatz. Uncertainties in measuring length and momentum are represented by the expressions (18) and (21), respectively. Their dependence on Δx_0 can be determined based on Δt. Both length and momentum quantitative estimations are relevant to physical systems where spacetime is likely curved and where energy is relativistic, especially in the presence of non-negligible gravitational fields, as in very early universe, neutron stars, and black holes.

Considering the extension of the Riemann manifold M_4 to the Finsler tangent bundle TM_8,[46–49] the quantum-induced revisions to the metric tensor are examined is Sec. 3.

3. Quantum-Induced Revisions to Fundamental Metric on Finsler Manifold

Caianiello proposed that the curvature in relativistic eight-dimensional spacetime tangent bundle $TM_8 = TM_4 \otimes M_4$ resembles the quantization of four-dimensional spacetime M_4.[61–64] The Riemann manifold considered for GR meets the latter condition. For the former condition, we adopt Finsler geometry, which can be viewed as Riemann geometry without the quadratic limitation on the length measure (Finsler structure or Minkowski smooth norm),

$$F^2(x, \dot{x}) = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{24}$$

The manifold M_4 with $F(x, \dot{x})$ is known as a Finsler space. The length of a curve on M_4, similar to the line element in GR, is expressed as $\int F^2(x, \dot{x})dt$, where $\dot{x} = dx/dt$ represents the relativistic four-velocity — the components of the contravariant vectors that are tangent to the manifold M_4 at the point x. A set of second-order ordinary differential equations describes the tangent bundle TM_4, which describes locally minimizing curves. Another noteworthy aspect is that, analogous to the relationship between Riemannian space and Euclidean space, the Finsler structure does not depend on x^μ; it can be associated with Minkowski space, particularly if the Finsler space allows for an x^μ-coordinate system. Given the homogeneity condition in $\dot{x}$, one finds that

- $F(x, \lambda\, \dot{x}) = \lambda\, F(x, \dot{x}), \quad \forall \lambda \in \mathbb{R}^+$,[47,65] and
- The ratio of lengths of any two collinear vectors does not depend on metric functions.

The first property, on one hand, enables the fulfillment of the requirements of doubly special relativity, loop quantum gravity, string theory, and noncommutative geometry — specifically, by extending coordinates within a continuous Finsler norm, from which one can derive the quantized or at the very least discretized metric tensor.

It is presumed that the most natural selection for λ would be $\Delta x^{\mu}_{\texttt{min}}$, Eq. (13). As elaborated in Sec. 2, $\Delta x^{\mu}_{\texttt{min}}$ imposes constraints on the spatial and momentum coordinates within GR, thereby establishing their uncertainties, which ensures that their measurements are neither arbitrary nor incoherent. Then, we find that

$$F\Big(x, \Big(\sqrt{-|g|}\,\hbar\sqrt{\beta_0}\Big)\,\dot{x}\Big) = (\sqrt{-|g|}\,\hbar\sqrt{\beta_0})\,F(x,\dot{x}), \quad \forall\Big(\sqrt{-|g|}\,\hbar\sqrt{\beta_0}\Big) \in \mathbb{R}^{+}. \tag{25}$$

On the other hand, the Finsler manifold has been suggested as the natural general space to explore the dispersion relation and the clock postulate of GR.[66] These are the reasons for the derivation of the modified metric tensor on Finsler manifold.[67]

A few remarks regarding the resulting Finsler structure, as presented in Eq. (25), are now appropriate, especially whether the quantum deformation preserves its smooth and continuous properties. The Finsler structure $F(x,\dot{x})$, where x denotes the coordinates and $\dot{x}$ signifies the tangents, or directions (such as that of velocities), is positively homogeneous of degree two. This implies that the resulting $F(x, \Delta x^{\mu}_{\texttt{min}}\,\dot{x})$ could be reexpressed as $\Delta x^{\mu}_{\texttt{min}}\,F(x,\dot{x})$, with $\Delta x^{\mu}_{\texttt{min}}$ being a positive scalar. Thus, the modified Finsler structure is also positively homogeneous with respect to $\Delta x^{\mu}_{\texttt{min}}\,\dot{x}$. This is apparent since $\dot{x}$ exclusively refers to directions rather than magnitudes or vectors, similar to the product $\Delta x^{\mu}_{\texttt{min}}\,\dot{x}$. However, the resulting $F(x, \Delta x^{\mu}_{\texttt{min}}\,\dot{x})$ seems to depart from being smooth and continuous due to the discretization introduced with $\Delta x^{\mu}_{\texttt{min}}\,\dot{x} = \pm\sqrt{-\texttt{det}g}\,\sqrt{\beta_0}\,\ell_{\texttt{pl}}\,\dot{x}$. The coefficient $\pm\sqrt{-\texttt{det}g}\,\sqrt{\beta_0}\,\ell_{\texttt{pl}}$ defines the roughness and discretization incorporated into the Finsler structure.

On TM_8, the metric tensor given as $g_{ab} = g_{\mu\nu}\otimes g_{\mu\nu}$ may be determined by the Hessian in the $(x,\dot{x})$-coordinates,

$$g_{ab}(x) = \frac{1}{2}\,\frac{\partial^2}{\partial\dot{x}^{\alpha}\,\partial\dot{x}^{\beta}}\;F^2(x, (\sqrt{-|g|}\hbar\sqrt{\beta})\dot{x}). \tag{26}$$

Given the second property that the ratios of collinear vectors are independent of the metric and the local coordinates on TM_8, which are composed as

$$x^a = \Big(x^{\alpha}, (\sqrt{-|g|}\hbar\sqrt{\beta})\dot{x}^{\alpha}\Big), \tag{27}$$

with $\dot{x}^{\alpha} = \partial x^{\alpha}/\partial\zeta^{\mu}$, the line element on TM_8 can be derived,[68,69]

$$d\tilde{s}^2 = g_{ab}(x)\,dx^a\,dx^b. \tag{28}$$

The counterpart line element in four-coordinate ζ^{μ} on M_4 is given as

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}\,d\zeta^{\mu}\,d\zeta^{\nu}. \tag{29}$$

In this regard, it is noteworthy to mention that the line element $d\tilde{s}^2$, whether in Eq. (28) or in Eq. (29), presumably provides an *accurate* measurement on both Finsler and Riemann manifolds. For instance, there is no probability distribution integrated, and no operator representation is considered. These features suggest that the measurement probability is assumed to be at its maximum. Apparently, this kind of measurement of the line element still reflects GR rather than QM conventional properties. With these remarks, we wanted to emphasize that this study merely proposes quantum-induced corrections, where certain approximations are still acknowledged. An entirely quantum representation of the line element measurement constitutes an essential aspect that should be worked out elsewhere. We conclude that the existing classical nature of the interconnected quantities — the line element ds^2, the fundamental tensor g, and the 1-form coordinate variations dx_μ — is what differentiates GR from QM. Should these be fully quantized appropriately, a unification of GR and QM could be achieved in the relativistic domain.

Then, by equating Eqs. (28) and (29), one obtains

$$\tilde{g}_{\mu\nu} = g_{ab}(x)\, \frac{\partial x^a}{\partial \zeta^\mu}\, \frac{\partial x^b}{\partial \zeta^\nu}. \tag{30}$$

The indexes $a, b \in \{0, 1, \ldots, 7\}$, while $\alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}$. The metric derived from a Finsler manifold obviously varies from that on Riemann manifold. It is not straightforwardly feasible to linearly convert the Finsler metric into the Riemann metric.[70] On the other hand, this might potentially be proposed by evaluating the line elements on both manifolds. We apply a recipe that may not be fully rigorous in a mathematical or physical prospective. Yet, as our approach still involves multiple approximations, it might be tolerated to apply the recipe with equating line elements on both Finsler and Riemann manifolds. Equation (30) represents the solution obtained by equating Eqs. (28) and (29) for the quantum-deformed metric, i.e. expressing the quantum-deformed metric in terms on Finsler metric and coordinate transformations. In light of the similarities identified between Eqs. (30) and (8), the suggested approach is thus grounded in both mathematical and physical reasoning.

Differentiating the coordinates on TM_8 with respect to ζ^μ, Eq. (30), leads to

$$\tilde{g}_{\mu\nu} = [1 + (-|g|\, \hbar^2\, \beta)|\ddot{x}|^2]\, g_{\mu\nu} = (1 + (-|g|\, \beta_0\, \ell_{\mathrm{p}}^2)\, |\ddot{x}|^2)\, g_{\mu\nu}, \tag{31}$$

where $\ddot{x}^\lambda = \partial \dot{x}^\lambda / \partial \zeta^\lambda$ and $|\ddot{x}|^2 \equiv \ddot{x}^\lambda \ddot{x}_\lambda = g_{\delta\gamma} \ddot{x}^\delta \ddot{x}^\gamma$ with λ, δ, γ as dummy indexes. It is obvious that both metrics are four-dimensional on Riemann manifold. The entire contributions of the generalized noncommutative Heisenberg algebra and thereby QM to $g_{\mu\nu}$ are summarized in the term

$$-|g|\, \beta_0\, \ell_p\, |\ddot{x}|^2\, g_{\mu\nu} = \mathcal{T} |\ddot{x}|^2\, g_{\mu\nu}, \tag{32}$$

so that the quantum-deformed fundamental metric can be reexpressed as

$$\tilde{g}_{\mu\nu}(x) = (1 + \mathcal{T}\, |\ddot{x}|^2)\, g_{\mu\nu}. \tag{33}$$

Utilizing this single extension $\mathcal{T}\, |\ddot{x}|^2$, Eq. (33), which was developed on the TM_8 manifold to integrate quantum deformations while maintaining the postulates of GR, an expression on the Riemann manifold is then derived. The torsion-free affine connections and metric compatibility, in addition to the nonmetricity of gravity, are entirely preserved in this work.

The quantum-induced revisions of the affine connections on Riemann manifold are presented in Sec. 4.

4. Quantum-Deformed Affine Connection on Riemann Manifold

Both $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ have common features. They map covariant tensors to contravariant tensors and vice versa. In their indexes μ and ν, they are symmetric, as their components are likely to remain the same. The symmetric properties are invariant under basis transformation, provided the indexes are of the same type, either contravariant or covariant. In the four-dimensional differentiable Lorentzian manifold M_4, both metric tensors are covariant, second-rank, and symmetric on M_4. For the affine connections, as expressed in Eq. (A.3), with the quantum-modified metric from Eq. (33), the partial derivatives are evidently commutative as well (Appendix A). In a free-falling frame, metric compatibility is likewise fulfilled; $\nabla_\sigma \tilde{g}_{\mu\nu} = \partial_\sigma \tilde{g}_{\mu\nu} = 0$. Thus, the quantum-induced corrections to the affine connections, Eq. (A.3), can be expressed as

$$\tilde{\Gamma}^{\,\gamma}_{\beta\mu} = \frac{1}{2}\tilde{g}^{\alpha\gamma}(\tilde{g}_{\alpha\beta,\mu} + \tilde{g}_{\alpha\mu,\beta} - \tilde{g}_{\beta\mu,\alpha}). \tag{34}$$

For curved space, by substituting Eq. (33) into Eq. (A.3), we obtain

$$\tilde{\Gamma}^{\,\gamma}_{\beta\mu} = \frac{1+2\mathcal{T}|\ddot{x}|^2}{1+\mathcal{T}|\ddot{x}|^2}\;\frac{1}{2}\;g^{\alpha\gamma}\big(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\big) = \frac{1+2\mathcal{T}|\ddot{x}|^2}{1+\mathcal{T}|\ddot{x}|^2}\Gamma^{\,\gamma}_{\beta\mu}. \tag{35}$$

It is clear that vanishing β_0 and/or $|\ddot{x}|^2$ straightforwardly retrieve the conventional affine connections. This is also the case at vanishing β, or vanishing minimal length uncertainty, or vanishing $|\ddot{x}|^2$: the cancellation of the additional Finsler geometric structure or the omission of quantization of GR. In Eq. (31), we have shown that all quantum-induced ingredients are interdependent. The minimal length uncertainty — that is, finite β — determines how to define the local coordinates on the Finsler TM_8, the components of the contravariant vectors tangent to the manifold, or the relativistic four-velocity $\dot{x}$. The parametrization of the four-coordinates on M_4 in eight-coordinates on TM_8 gives rise to the second-order derivatives of the tangent covectors $|\ddot{x}|^2$ and creates additional geometric structures. Equation (35) reveals that the corrections added to the affine connections are exclusively compacted into the coefficient $(1+2\mathcal{T}|\ddot{x}|^2)/(1+\mathcal{T}|\ddot{x}|^2)$. This means that the affine (linear) connections preserve, on theone hand, its geometric nature as in GR, for instance. On the other hand, the quantum-induced corrections via additional curvatures emerged on a higher-dimensional manifold reveal additional geometric structures, especially at relativistic scales at which $\mathcal{T}|\ddot{x}|^2$ becomes significant. An analogous picture may be borrowed from the radiation beam: in Newtonian mechanics, this beam is a continuous stream, whereas in QM this nature is not only preserved but discontinuity, for instance, is also revealed.

5. Parallel Transport on Riemann Manifold

In a flat space, covariant derivatives, which involve the differentiation of both vector components and basis vectors, reduce to ordinary derivatives. In a curved space, however, the differentiation of the basis vectors can be expressed using Christoffel symbols. In both flat and curved spaces, covariant derivatives can be defined as the rates of change of tangent vector fields (ordinary derivatives, for example) with the normal component subtracted; this process is known as parallel transport.[71] If the covariant derivatives of a vector $\vec{v} = v^\alpha e_\alpha$ vanish, it indicates that $\vec{v}$ is parallel-transported, meaning that $\vec{v}$ is maintained as constant as possible:

$$\frac{\partial}{\partial\lambda} v^\alpha = -v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho}. \tag{36}$$

It is obvious that the parallel transport does not depend on the connections $\Gamma^{\,\alpha}_{\sigma\rho}$. With the inclusion of quantum-induced corrections, $\Gamma^{\,\alpha}_{\sigma\rho}$ is to be replaced with $\tilde{\Gamma}^{\,\alpha}_{\sigma\rho}$, Eq. (35); thus Eq. (36) can be rewritten as

$$\frac{\partial}{\partial\lambda}\,\tilde{v}^\alpha = -\frac{1}{1+\mathcal{T}|\ddot{x}|^2}\,v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho} - \frac{2\mathcal{T}|\ddot{x}|^2}{1+\mathcal{T}|\ddot{x}|^2}\,v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho}. \tag{37}$$

Let us define the affine connections associated with the vector $\vec{v}$ using the parametrization λ, and examine the scenario where the vector $vecv(\lambda)$ is parallel-transported to $\lambda + d\lambda$, yielding $\vec{v}(\lambda + d\lambda)$. The partial derivative (change) can be substituted with a limit as $d\lambda \to 0$:

$$\lim_{d\lambda\to 0}\frac{v^\alpha(\lambda+d\lambda) - v^\alpha(\lambda)}{d\lambda} \simeq -\frac{1}{1+\mathcal{T}|\ddot{x}|^2}\,v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho} - \frac{2\mathcal{T}|\ddot{x}|^2}{1+\mathcal{T}|\ddot{x}|^2}\,v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho}. \tag{38}$$

Following this, the change in both vectors $\vec{v}(\lambda) = v^\alpha(\lambda)e_\alpha$ and $\vec{v}(\lambda + d\lambda) = v^\alpha(\lambda + d\lambda)e_\alpha$ due to parallel transport is represented in terms of components as

$$\delta\tilde{v}^\alpha \simeq -\left(\frac{1}{1+\mathcal{T}|\ddot{x}|^2} + \frac{2\mathcal{T}|\ddot{x}|^2}{1+\mathcal{T}|\ddot{x}|^2}\right) v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho}\,d\lambda. \tag{39}$$

The corresponding vector formulation leads to $\delta\tilde{\vec{v}} = \delta\tilde{v}^\alpha e_\alpha$. The vector form can be expressed as follows:

$$\vec{v}(\lambda + d\lambda) \simeq \vec{v}(\lambda) - \left(\frac{1}{1+\mathcal{T}|\ddot{x}|^2} + \frac{2\mathcal{T}|\ddot{x}|^2}{1+\mathcal{T}|\ddot{x}|^2}\right) v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho}\,d\lambda\,e_\alpha. \tag{40}$$

Similar to Eq. (36), both terms on the right-hand side of Eq. (39) express affine connections with quantum-induced corrections. For vanishing β_0 and/or $|\ddot{x}|^2$, $\delta\vec{v}$ is then reduced to Eq. (36),

$$\delta v^\alpha \simeq -v^\rho\,\Gamma^{\,\alpha}_{\sigma\rho}\,d\lambda. \tag{41}$$

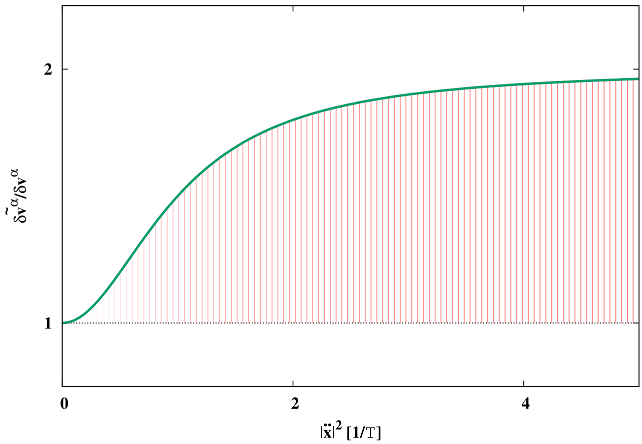

Fig. 1. The dependence of $\delta\tilde{v}^\alpha$ normalized to δv^α on $|\ddot{x}|^2$ given in units of $1/\mathcal{T}$. The dashed area represents the quantum-induced corrections.

Equation (39) reveals that the entire contributions induced by the geometric quantum revisions are given as follows:

$$\frac{1}{1+\mathcal{T}|\ddot{x}|^2} + \frac{2\mathcal{T}|\ddot{x}|^2}{1+\mathcal{T}|\ddot{x}|^2}\,. \tag{42}$$

These contributions are illustrated as the dashed region in Fig. 1. To prevent any controversial debate regarding the physical interpretation of $|\ddot{x}|^2$, we standardize $|\ddot{x}|^2$ to $\mathcal{T}$. In $\mathcal{T}$, the proposed geometric quantum corrections, Eq. (32), are incorporated. The horizontal dotted line in Fig. 1 serves as the reference point where $\delta\tilde{v}^\alpha = \delta v^\alpha$. From the correlation of the normalized $\delta\tilde{v}^\alpha/\delta v^\alpha$ with $|\ddot{x}|^2/\mathcal{T}$, it is possible to evaluate the contributions that arise from the minimal-length discretization. In the case of small $|\ddot{x}|^2$, there is a gradual increase that resembles the behavior of a tanh-function. However, for large $|\ddot{x}|^2$, $\delta\tilde{v}^\alpha/\delta v^\alpha$ approaches an asymptotic value, specifically a doubling.

In Sec. 6, the symmetry characteristics of the revisions induced by quantum effects on the affine connections are analyzed.

6. Symmetry Properties of Quantum-Deformed Affine Connections

It is obvious that the symmetric property of the affine connections depends on (i) the symmetric property of the metric tensor, and (ii) the commutativity of the partial derivatives. The quantum-induced revision of the affine connections, Eq. (39), seems to fulfill two conditions.

- In any coordinate system, the quantum-induced revisited affine connections can solely be represented through the metric tensor and its derivatives,

$$\tilde{\Gamma}^{\,\gamma}_{\beta\mu} = \frac{1}{2}\tilde{g}^{\alpha\gamma}(\tilde{g}_{\alpha\beta,\mu} + \tilde{g}_{\alpha\mu,\beta} - \tilde{g}_{\beta\mu,\alpha}), \tag{43}$$

where the quantum-deformed metric $\tilde{g}$ is symmetric[35]; consequently, the quantum-induced revisited affine connections are symmetric as well.

- Let us express the affine connections as[72]

$$\tilde{\Gamma}^{\gamma}_{\ \beta\mu} = \frac{\partial x^{\gamma}}{\partial X^{\alpha}} \frac{\partial^2 X^{\alpha}}{\partial x^{\beta} \partial x^{\mu}}, \tag{44}$$

where x^{λ} and X^{α} represent different coordinates in curved space. The commutativity of the partial derivatives continues to hold in the quantum-induced revisited affine connections, as shown in Eq. (39). This principle remains applicable even when X^{α} is adjusted to incorporate the presence of a minimal length uncertainty.

Therefore, we conclude that the quantum-induced revisited affine connections are symmetric in their lower indexes $\tilde{\Gamma}^{\gamma}_{\ (\beta\mu)}$, so that

$$\tilde{\Gamma}^{\gamma}_{\ (\beta\mu)} = \tilde{\Gamma}^{\gamma}_{\ \beta\mu} = \tilde{\Gamma}^{\gamma}_{\ \mu\beta}. \tag{45}$$

Also, because the quantum-deformed affine connections on the Riemann manifold are torsion-free,

$$T^{\gamma}_{\beta\mu} = \tilde{\Gamma}^{\gamma}_{\ \beta\mu} - \tilde{\Gamma}^{\gamma}_{\ \mu\beta} = 2\tilde{\Gamma}^{\gamma}_{\ [\beta\mu]} = 0, \tag{46}$$

where $\tilde{\Gamma}^{\gamma}_{\ [\beta\mu]} = 0$.

7. Conclusions and Outlook

The minimal-length framework that arises from noncommutative Heisenberg algebra and GUP is hypothesized to merge gravity into QM by modifying the HUP to include the effects of gravitational fields.[25,27] When this methodology is utilized in the context of GR, the fundamental metric, for instance, gains additional contributions that are functions of the GUP parameter β_0, the second-order derivatives of tangent covectors $|\ddot{x}|^2$, along with the Planck constant $\hbar$ and length $\ell_p = \sqrt{G\hbar/c^3}$. To implement quantum-induced revisions on the spacetime manifold M_4, we have followed the recipe that additional curvatures emerge in the relativistic eight-dimensional spacetime tangent bundle $TM_8 = TM_4 \otimes M_4$ — the Finsler manifold — and they mimic the quantization on the four-dimensional spacetime M_4, which is a Riemann manifold. This is a differential manifold M_4 that is furnished with a tangent bundle manifold TM_4, in which the limitations on the nonquadratic length measurement for vectors are eased. The local coordinates x^{μ} on M_4 are combined with the tangent vectors $\dot{x}^{\alpha} = dx^{\alpha}/d\zeta^{\mu}$ on TM_4. Accordingly, the metric is determined by the Hessian in $(x^{\alpha}, \Delta x^{\mu}_{\mathtt{min}} \dot{x}^{\alpha})$-coordinates on TM_8. We remark that $\Delta x^{\mu}_{\mathtt{min}}$ integrates various quantum-mechanical properties afforded to the GR spacetime and momentum, as outlined in Sec. 2. On the other hand, the coordinates and tangent covectors on Riemann and Finsler manifolds, respectively, are not yet properly quantized.

The scalar quantity $\mathcal{T}|\ddot{x}|^2$ is presumed to combine the minimal measurable length $\Delta x_{\mathtt{min}}$ with GR, QM, and geometric quantities. This is the single extension introduced to the fundamental metric. On Riemann manifold, torsion-free and metric-compatible affine connections, as well as the nonmetricity of gravity, remain unchanged. The results obtained in this work, if correct, suggest quantum-induced corrections to GR, especially in the relativistic regime. This work focuses on the quantum-induced corrections to the fundamental metric and affine connections. We have discussed the symmetric property and evaluated the dependence of a normalized parallel-transported vector on the second-order derivatives of tangent covectors $|\ddot{x}|^2$, which exhibits a slow tanh-like increase at small $|\ddot{x}|^2$. At large $|\ddot{x}|^2$, it reaches an asymptotic value: a doubling. This observation demonstrates that the minimal measurable length uncertainty and the quantization procedure are significant, especially in the relativistic regime where $\mathcal{T}|\ddot{x}|^2$ is finite.

We conclude that the quantum-induced corrections to the affine connections are exclusively factorized into the coefficient $(1+2\mathcal{T}|\ddot{x}|^2)/(1+\mathcal{T}|\ddot{x}|^2)$, which combines quantum-deformed, geometric-structural, noncommutative-algebraic, and gravitational ingredients. On one hand, this means that the affine connections preserve all properties of their unquantized counterparts, such as being torsion-free and metric-compatible. On the other hand, their geometric nature as connecting nearby tangent spaces on a *smooth* manifold is also preserved on *discrete* spaces. The quantization via an additional curvature on a higher-dimensional manifold likely reveals fine geometric structure, analogous to the radiation beam in classical and QM.

We have studied how the quantum-induced revisited affine connections perform parallel transport of a tangent vector on Riemann manifold. Depending on $|\ddot{x}|^2/\mathcal{T}$, the quantum-induced corrections can nearly double the parallel transport of the unquantized affine connections. We conclude that this study suggests quantum-mechanical corrections to

- The fundamental metric $g_{\mu\nu}$, Eq. (33), and
- The affine connections $\Gamma^{\gamma}_{\beta\mu}$, Eq. (35).

These corrections are added linearly to $g_{\mu\nu}$ and $\Gamma^{\gamma}_{\beta\mu}$ so that they diminish or grow depending on the energy regime. The latter dictates when β_0 and $|\ddot{x}|^2$ or the entire corrections $\mathcal{T}$ vanish or increase. As outlined in Sec. 2, the RGUP approach utilized in this study reveals additional features of GR in the relativistic regime:

- Spacetime discretization with an inaccessible spacetime element $\Delta x^{\mu}_{\mathtt{min}}$,
- Noncommutative relations between pairs of physical quantities, and
- Uncertainties and incoherent measurements in coordinates and momenta, for instance.

On the other hand, we still need to quantize the line element and propose noncommutation relations between the metric tensor and the 1-form coordinate variations before drawing any conclusion about a full reconciliation of the principles of GR and QM.

Appendix A. Differential Geometry and Affine Connections

Affine (linear) connections are identified as geometric structures that associate nearby tangent (curved) spaces; they allow for the differentiability of tangent vector fields or provide them with a restricted dependence on the manifold within a particular vector space.[72] Moreover, they function as mappings that assign a covariant derivative or a new tangent vector to every tangent vector and vector field. In differential geometry, the general form of the affine connections was suggested as[73]

$$\Gamma^{\mu}_{\lambda\nu} = \{^{\mu}_{\lambda\nu}\} + K^{\mu}_{\lambda\nu} + \frac{1}{2}(Q^{\mu}_{\lambda\nu.} + Q^{\mu}_{\nu\lambda.} - Q^{\mu}_{.\nu\lambda}), \tag{A.1}$$

where the dot in lower indexes refers to the position of the upper index, $\{^{\mu}_{\lambda\nu}\}$ is the Christoffel symbol, and $Q_{\mu\nu\lambda} = -D_{\mu}(\Gamma)g_{\nu\lambda}$ is the covariant derivative of the metric tensor. $K^{\mu}_{\lambda\nu} = \frac{1}{2}(T^{\mu}_{.\lambda\,\nu} - T^{\mu}_{\lambda.\nu} - T^{\mu}_{\nu.\lambda})$ is the contortion known as the Levi–Civita connection, and $T^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\lambda\nu} - \Gamma^{\mu}_{\nu\lambda} = 2\Gamma^{\mu}_{[\lambda\nu]}$ gives the torsion. The latter represents the antisymmetric part of the connection. GR assumes metric compatibility of the affine connections, which implies linear independence of the partial-derivative tangent vectors and accordingly leads to vanishing $D_{\mu}(\Gamma)g_{\nu\lambda}$. Also, the metric compatibility of the connection naturally arises if the covariant derivative is a tensor applying the Leibniz rule.[74] Metric compatibility indicates that a flat space can be identified locally within an appropriate frame (Minkowski space). In a free-falling frame, for example, $g_{\nu\lambda} = \eta_{\nu\lambda}$; then $D_{\mu}(\Gamma)g_{\nu\lambda}$ vanishes for $g_{\nu\lambda} = \eta_{\nu\lambda}$.[72] In such a frame, the covariant derivative of a tensor is the same for all observers and frames; that is, $D_{\mu}(\Gamma)g_{\nu\lambda} = 0$.[73–76]

Furthermore, GR assumes that the affine geodesics match the metrical geodesics.[a] The latter is clearly given by extremizing ds^2, the spacetime interval.[34,35] For the torsion-free assumption, $K^{\mu}_{\lambda\nu}$ entirely vanishes, the metric functions as the gravitational field potential, and Riemann geometry is symmetric (the energy–momentum tensor is also symmetric). Then, the affine connection reduces to

$$\Gamma^{\mu}_{\lambda\nu} = \{^{\mu}_{\lambda\nu}\}. \tag{A.2}$$

The assumption of symmetric connection coefficients leads to commutative partial derivatives, Eq. (44). Under the conditions of metric compatibility, symmetry of the metric tensor indexes, and partial-derivative commutation, there exists one particular version of the connection coefficients (Levi-Civita connection). Then, the Christoffel symbols can be expressed as[75]

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}), \tag{A.3}$$

where $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$.

[a] There are other assumptions, such as nonsymmetric energy–momentum tensor or finite torque density as in Einstein–Cartan–Sciama–Kibble theory.[77]

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